Integration Bee Practice Solutions

1. $\int \left(\frac{\sqrt{x}}{2} - \frac{\sqrt{2}}{x}\right) dx = \frac{1}{2}\int x^{1/2} dx - \sqrt{2}\int \frac{1}{x} dx$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x^{3/2}}{3} - \sqrt{2} \ln|x| + C = \boxed{\frac{x^{3/2}}{3} - \sqrt{2}\ln|x| + C}$

2. $\int 7 \sin \frac{\theta}{3} d\theta$ Let $u = \frac{\theta}{3}$ $du = \frac{1}{3} d\theta$

So, $3du = d\theta$

$7 \cdot 3 \int \sin u \, du$

$= 21(-\cos u) + C = -21\cos\frac{\theta}{3} + C$

3. $\int -3\csc^2 x \, dx = -3\int \csc^2 x \, dx$

$= -3 \cdot -\cot x + C$

$= 3\cot x + C$

4. $\int (1 + \tan^2\theta) d\theta = \int \sec^2\theta \, d\theta = \tan\theta + C$

5. $\int \sqrt{4y-1} \, dy \longrightarrow \frac{1}{4}\int \sqrt{u} \, du = \frac{1}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$

Let $u = 4y - 1$

$du = 4dy \Rightarrow \frac{1}{4} du = dy$

$= \boxed{\frac{1}{6}(4y-1)^{3/2} + C}$

6. $\int \frac{3\,dx}{(2-x)^2}$ Let $u = 2-x$
$du = -dx$
$-du = dx$

$-3\int \frac{1}{u^2}du = -3\int u^{-2}du = \frac{3}{u} + C$

$= \boxed{\frac{3}{2-x} + C}$

7. $\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx$

Let $u = \sin\frac{x}{3}$

$du = \cos\frac{x}{3}\cdot\frac{1}{3}dx$

$3du = \cos\frac{x}{3}\,dx$

$3\int u^5 du$

$= \frac{\cancel{3}u^6}{\cancel{6}_2} + C$

$= \boxed{\frac{1}{2}\sin^6\frac{x}{3} + C}$

8. $\int \frac{\cos(\sqrt{x}+3)dx}{\sqrt{x}}$

Let $u = \sqrt{x}+3$

$du = \frac{1}{2}x^{-1/2}dx$

$2du = \frac{1}{\sqrt{x}}dx$

$\downarrow$

$2\int \cos u \, du$

$= 2\sin u + C = \boxed{2\sin(\sqrt{x}+3)+C}$

9. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

Let $u = \frac{r^3}{18}-1$

$du = \frac{\cancel{3}r^2}{\cancel{18}_6}dr$

$\downarrow$

$6\int u^5 du$

$6du = r^2 dr$

$= \cancel{6}\frac{u^6}{\cancel{6}} + C = \boxed{\left(\frac{r^3}{18}-1\right)^6 + C}$

10. $\int \frac{7x^9 - 3x^2 + 4}{2x^5} dx$

$= \int\left(\frac{7x^9}{2x^5} - \frac{3x^2}{2x^5} + \frac{4}{2x^5}\right)dx = \int\left(\frac{7}{2}x^4 - \frac{3}{2x^3} + \frac{4}{2x^5}\right)dx$

$= \frac{7}{2}\cdot\frac{x^5}{5} - \frac{3}{2}\cdot\frac{-1}{2x^2} + \frac{4}{2}\cdot\frac{-1}{4x^4} = \boxed{\frac{7x^5}{10} + \frac{3}{4x^2} - \frac{1}{2x^4} + C}$

11. $\int \frac{\sin(2x+1)}{\cos^2(2x+1)}\,dx$ Let $u = \cos(2x+1)$

$du = -\sin(2x+1)\cdot 2dx$

$-\frac{1}{2}du = \sin(2x+1)dx$

$$\downarrow$$

$$-\frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2}\cdot\frac{-1}{u} + C = \boxed{\frac{1}{2\cos(2x+1)} + C}$$

12. $\int 2\sin x\cos x\,dx$ → double angle formula

$2\sin x\cos x = \sin 2x$

$$= \int \sin 2x\,dx = \boxed{-\frac{1}{2}\cos 2x + C}$$

13. $\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \sqrt{\frac{x-1}{x^4\cdot x}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx$

$= \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx$ Let $u = \frac{1}{x}$ $du = -\frac{1}{x^2}dx$

$\Rightarrow -\int \sqrt{1-u}\,du$ Let $w = 1-u$ $dw = -du$

$\Rightarrow \int \sqrt{w}\,dw = \frac{2}{3}w^{3/2} + C = \frac{2}{3}(1-u)^{3/2} + C$

$$= \boxed{\frac{2}{3}\left(1-\frac{1}{x}\right)^{3/2} + C}$$

14. $\displaystyle\int \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$ Let $u = 1+\sqrt{y}$

$du = \dfrac{1}{2\sqrt{y}}\,dy$

$\downarrow$

$\displaystyle\int \frac{1}{u^2}\,du = -\frac{1}{u} + C = \boxed{\frac{-1}{1+\sqrt{y}} + C}$

15. $\displaystyle\int \frac{\sin w}{(3+2\cos w)^2}\,dw$ Let $u = 3+2\cos w$

$du = -2\sin w\,dw$

$-\dfrac{1}{2}\,du = \sin w\,dw$

$\downarrow$

$\displaystyle -\frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2}\cdot -\frac{1}{u} + C = \boxed{\frac{1}{2(3+2\cos w)} + C}$

16. $\displaystyle\int 2x e^{x^2} \cos(e^{x^2})\,dx$ Let $u = e^{x^2}$

$du = e^{x^2}\cdot 2x\,dx$

$\downarrow$

$\displaystyle\int \cos u\,du = \sin u + C = \boxed{\sin(e^{x^2}) + C}$

17. $\int x^3 \cos(x^4-1)\sin^8(x^4-1)\,dx$

Let $u = \sin(x^4-1)$

$du = \cos(x^4-1)\cdot 4x^3\,dx$

$\Rightarrow \frac{1}{4}du = x^3\cos(x^4-1)\,dx$

$$\frac{1}{4}\int u^8\,du = \frac{1}{4}\cdot\frac{u^9}{9} + C = \boxed{\frac{1}{36}\sin^9(x^4-1)+C}$$

18. $\int \cos^6 x \sin^3 x\,dx$ Let $u = \cos x$

$du = -\sin x\,dx$

$$\int \underbrace{\cos^6 x}_{u^6} \cdot \underbrace{\sin^2 x}_{\substack{1-\cos^2 x \\ \text{or } 1-u^2}} \cdot \underbrace{\sin x\,dx}_{-du}$$

$$= \int u^6(1-u^2)(-du) = \int -u^6(1-u^2)\,du$$

$$= \int(-u^6+u^8)\,du = -\frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \boxed{-\frac{\cos^7 x}{7} + \frac{\cos^9 x}{9} + C}$$

19. $\int\left(e^{3x} - \frac{5}{e^x}\right)dx = \int\left(e^{3x} - 5e^{-x}\right)dx$

$= \frac{e^{3x}}{3} - \frac{5e^{-x}}{-1} + C = \boxed{\frac{e^{3x}}{3} + \frac{5}{e^x} + C}$

20. $\int \frac{\sec y \tan y \, dy}{2 + \sec y}$ Let $u = 2 + \sec y$
$du = \sec y \tan y \, dy$

$\int \frac{1}{u} \, du = \ln|u| + C = \boxed{\ln|2 + \sec y| + C}$

21. $\int \frac{3}{3x-2} \, dx$ Let $u = 3x - 2$
$du = 3 \, dx$

$\int \frac{1}{u} \, du = \ln|u| + C = \boxed{\ln|3x-2| + C}$

22. $\int \frac{dx}{x(\ln x)^2}$ Let $u = \ln x$
$du = \frac{1}{x} dx$

$\int \frac{1}{u^2} \, du = \frac{-1}{u} + C = \boxed{\frac{-1}{\ln x} + C}$

23. $\displaystyle\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$ $\quad$ Let $u=\ln(\sec x+\tan x)$

$du = \dfrac{1}{\sec x+\tan x}\cdot \sec x\tan x+\sec^2 x\,dx$

$du = \dfrac{1}{\cancel{\sec x+\tan x}}\sec x(\cancel{\tan x+\sec x})dx$

$\downarrow$

$\displaystyle\int \frac{1}{\sqrt{u}}\,du$

$\displaystyle = \int u^{-1/2}\,du = 2\sqrt{u}+C = \boxed{2\sqrt{\ln(\sec x+\tan x)}+C}$

24. $\displaystyle\int x\cos x\,dx$ $\quad$ Let $u=x$ $\quad$ $dv=\cos x\,dx$

$du=dx$ $\quad$ $v=\sin x$

$\downarrow$

$\displaystyle x\sin x - \int \sin x\,dx = \boxed{x\sin x+\cos x+C}$

25. $\displaystyle\int x^3e^{7x}\,dx$

use integ by parts 3 times or use the tabular method

Deriv		Integ
x^3	$+$	e^{7x}
$3x^2$	$-$	$\frac{1}{7}e^{7x}$
$6x$	$+$	$\frac{1}{49}e^{7x}$
6	$-$	$\frac{1}{343}e^{7x}$
0		$\frac{1}{2401}e^{7x}$

$$\frac{1}{7}x^3e^{7x} - \frac{3}{49}x^2e^{7x} + \frac{6}{343}xe^{7x} - \frac{6}{2401}e^{7x} + C$$

26. $\int e^x \sin x\,dx$ Let $u = \sin x$ $dv = e^x dx$
$du = \cos x\,dx$ $v = e^x$

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$ use parts again

$u = \cos x$ $dv = e^x dx$
$du = -\sin x\,dx$ $v = e^x$

$\int e^x \sin x\,dx = e^x \sin x - \left[e^x \cos x - \int -e^x \sin x\,dx \right]$

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \cancel{\int e^x \sin x\,dx}$$
$$+\int e^x \sin x\,dx \qquad\qquad + \cancel{\int e^x \sin x\,dx}$$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x\,dx = \boxed{\frac{e^x \sin x - e^x \cos x}{2} + C}$$

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27. $\int \ln x\,dx$ Let $u = \ln x$ $dv = dx$
$du = \frac{1}{x}dx$ $v = x$

$$= x\ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln x - \int dx = \boxed{x\ln x - x + C}$$

28. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

decompose the integrand using partial fractions

$$\frac{x^2+2x-1}{x(2x^2+3x-2)} = \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$x=0$ $\quad -1 = A(-1)(2) \Rightarrow -1 = -2A \Rightarrow A = 1/2$

$x=-2$ $\quad -1 = C(-2)(-5) \Rightarrow -1 = 10C \Rightarrow C = -1/10$

$x=1/2$ $\quad 1/4 = B(1/2)(5/2) \Rightarrow 1/4 = 5/4 B \Rightarrow B = 1/5$

$$\int \left(\frac{1/2}{x} + \frac{1/5}{2x-1} - \frac{1/10}{x+2}\right) dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

29. $\int \frac{4x^2-7}{2x+3}\,dx$ Since the degree of the numerator is greater than the degree of the denom, we must use long division.

$$\begin{array}{r} 2x-3 \\ 2x+3\,\overline{)\,4x^2+0x-7} \\ \ominus 4x^2 \ominus 6x \\ \hline -6x-7 \\ \oplus 6x \oplus 9 \\ \hline 2 \end{array}$$

$$\int\left(2x-3+\frac{2}{2x+3}\right)dx$$

$$= 2\frac{x^2}{2} - 3x + 2\,\frac{\ln|2x+3|}{2} + C$$

$$= \boxed{x^2 - 3x + \ln|2x+3| + C}$$

30. $\int \frac{2\,dx}{x^2-6x+10}$ ← since the denom doesn't factor nicely, we complete the square

$$\int \frac{2\,dx}{(x^2-6x+9)+10-9} = \int \frac{2\,dx}{(x-3)^2+1}$$

Let $u=x-3$
$du=dx$

$$\int \frac{2\,du}{u^2+1} = 2\arctan u + C$$
$$= \boxed{2\arctan(x-3)+C}$$

31. $\int \frac{x}{x-3}\,dx$ You could use long division here or use another way.

Let $u=x-3 \Rightarrow u+3=x$
$du=dx$

$$\int \frac{u+3}{u}\,du = \int\left(1+\frac{3}{u}\right)du = u+3\ln|u|$$

$$= \boxed{x-3+3\ln|x-3|+C}$$

32. $\int \cos(\ln x)\,dx$ Let $u=\cos(\ln x)$ $dv=dx$

$du=\dfrac{-\sin(\ln x)}{x}dx$ $v=x$

$$\int \cos(\ln x)\,dx = x\cos(\ln x) - \int \frac{-x\sin(\ln x)}{x}dx$$

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$$

Let $u=\sin(\ln x)$ $dv=dx$

$du=\dfrac{\cos(\ln x)}{x}dx$ $v=x$

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + \left(x\sin(\ln x) - \int \frac{x\cos(\ln x)}{x}dx\right)$$

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \cancel{\int \cos(\ln x)\,dx}$$

$$+\int \cos(\ln x)\,dx \qquad\qquad + \cancel{\int \cos(\ln x)\,dx}$$

$$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$$

$$\int \cos(\ln x)\,dx = \boxed{\frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C}$$

33. $\int \frac{x+5}{x^2+x-2}\,dx$ decompose using the method of partial fractions

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$x+5 = A(x-1) + B(x+2)$$

$x=1$: $6 = 3B \Rightarrow B=2$

$x=-2$: $3 = A(-3) \Rightarrow A=-1$

$$\int \left(\frac{-1}{x+2} + \frac{2}{x-1} \right) dx$$

$$= \boxed{-\ln|x+2| + 2\ln|x-1| + C}$$

34. $\int \frac{\cos x}{\sin x+1}\,dx$ Let $u = \sin x + 1$
$du = \cos x\,dx$

$$\int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sin x+1| + C}$$

35. $\int x^3\sqrt{x^2+4}\,dx$ use the trig sub
$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$

$$\int (2\tan\theta)^3\sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta$$

$$= \int 8\tan^3\theta\sqrt{4\sec^2\theta}\cdot 2\sec^2\theta\,d\theta$$

$$= \int 8\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta$$

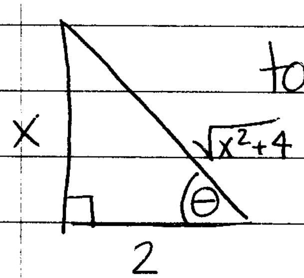
$$= 32\int \tan^3\theta\sec^3\theta\,d\theta$$

Let $u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$

$$= 32\int \underbrace{\tan^2\theta}_{\substack{\sec^2\theta-1\\ =u^2-1}}\cdot\underbrace{\sec^2\theta}_{u^2}\cdot\underbrace{\sec\theta\tan\theta\,d\theta}_{du}$$

$$= 32\int (u^2-1)u^2\,du = 32\int(u^4-u^2)\,du$$

$$= 32\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C = \frac{32\sec^5\theta}{5} - \frac{32\sec^3\theta}{3}+C$$



$\tan\theta = x/2$

$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

$$\boxed{\frac{32}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 + C}$$

36. $$\int \frac{\ln(\cos x)}{\cot x}\,dx$$

Let $u = \ln(\cos x)$

$$du = \frac{1}{\cos x} \cdot -\sin x\,dx$$

$$du = -\tan x\,dx$$

$$-du = \tan x\,dx$$

$$\int \frac{\ln(\cos x)}{\cot x}\,dx = \int \tan x \cdot \ln(\cos x)\,dx$$

$$= -\int u\,du = -\frac{u^2}{2} + C$$

$$= \boxed{-\frac{(\ln(\cos x))^2}{2} + C}$$